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$$KQ = \frac{+15 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}} - \beta}{16},$$

$$LQ = \frac{+15 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}} + \beta}{16},$$

$$LN = \sqrt{AL \cdot LQ} = \frac{1}{8} \sqrt{+34 + 2\sqrt{17} + 2\sqrt{34 + 2\sqrt{17}} - 2\beta'},$$

$$KM = \sqrt{AK \cdot KQ} = \frac{1}{8} \sqrt{+34 + 2\sqrt{17} + 2\sqrt{34 + 2\sqrt{17}} + 2\beta'},$$

$$[\beta' = \sqrt{2(\sqrt{17} - 3)(2\sqrt{17} - \sqrt{34 + 2\sqrt{17}})}; \text{ and } \beta \times (-1 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}}) = 4 \cdot \beta'],$$

$$LN \times KM = \frac{1}{64} \sqrt{2(\sqrt{17} + 3)(2\sqrt{17} + \sqrt{34 - 2\sqrt{17}})} = \frac{4\alpha}{64},$$

$$NM^2 = LK^2 + (KM - LN)^2$$

$$= 4 \cdot EH^2 + KM^2 + LN^2 - \frac{8\alpha}{64}$$

$$= \frac{+136 - 8\sqrt{17} + 8\sqrt{34 - 2\sqrt{17}} - 8\alpha}{64},$$

$$[\alpha' = \sqrt{2(\sqrt{17} + 3)(2\sqrt{17} - \sqrt{34 - 2\sqrt{17}})}; \text{ and } 4\alpha = \alpha' \times (-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}})],$$

$$\therefore NM^2 = \frac{+136 - 8\sqrt{17} + 8\sqrt{34 - 2\sqrt{17}} - 2(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}})\alpha'}{64}.$$

And, in the circle with radius equal to unity,  $NM$  represents the value  $2 \sin(2\pi/17) \times 1$ ; and therefore

$$\begin{aligned} 4 \cos^2 \frac{2\pi}{17} &= 4 - NM^2 \\ &= \frac{+120 + 8\sqrt{17} - 8\sqrt{34 - 2\sqrt{17}} + 2(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}})\alpha'}{64} \\ &= \left[ \frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + \alpha'}{8} \right]^2 \end{aligned}$$

and

$$2 \cos \frac{2\pi}{17} = \frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + \sqrt{2(\sqrt{17} + 3)(2\sqrt{17} - \sqrt{34 - 2\sqrt{17}})}}{8}.$$

And this value of  $2 \cos(2\pi/17)$  will be found to agree with that given in Klein's *Famous Problems in Elementary Geometry* (Beman & Smith), though our  $\alpha'$  is there written in a different form.

## II. HISTORICAL NOTE BY R. C. ARCHIBALD, Brown University.

This method of construction is due to H. W. Richmond, *Quarterly Journal of Mathematics*, Volume 26, 1893, pp. 206–207; and *Mathematische Annalen*, Volume 67, 1909, pp. 460–461. It is reproduced on page 34 of H. P. Hudson's *Ruler and Compasses*, London, 1916.

Various constructions of the regular polygon of seventeen sides were reviewed by R. Goldenring in his *Die elementargeometrischen Konstruktionen des regelmässigen Siebzehnecks* (Leipzig, 1915), but many omissions in this professedly complete survey were noted by the writer in the *Bulletin of the American Mathematical Society*, vol. 22, 239–246. The first solution in an English publication, given by Lowry in 1819,<sup>1</sup> was reproduced in this *Monthly*<sup>2</sup> in 1899 and 1914. Other solutions and historical notes are set forth in the articles printed above, pages 322–326.

**2767 [1919, 171]. Proposed by W. W. JOHNSON, U. S. Naval Academy.**

Let the complex quantities  $p$ ,  $q$ , and  $r$  satisfy the relation  $p^2 + q^2 + r^2 = 0$ ; prove that the corresponding vectors  $OP$ ,  $OQ$ , and  $OR$  are such that if any two of them are taken as conjugate semi-diameters of an ellipse, the third lies on the minor axis, and its length is the distance from the center to either focus.

**SOLUTION BY A. PELLETIER, Montreal, Can.**

Let  $(x^2/a^2) + (y^2/b^2) = 1$ , be the equation of the ellipse having  $OP$  and  $OQ$  for conjugate semi-diameters ( $2a$  and  $2b$  being the axes, and  $a \geqq b$ ). If  $\alpha$ ,  $\alpha'$ ,  $\alpha''$  are the respective arguments of

<sup>1</sup> *The Mathematical Repository*, new series, vol. 4, p. 160; Lowry's proof occupies pages 160–168.

<sup>2</sup> Volume 6, p. 239 and volume 21, p. 252.

$OP$ ,  $OQ$ , and  $OR$ , we have

$$OP^2(\cos 2\alpha + i \sin 2\alpha) + OQ^2(\cos 2\alpha' + i \sin 2\alpha') = - OR^2(\cos 2\alpha'' + i \sin 2\alpha''),$$

from datum; hence,

$$OP^2 \cos 2\alpha + OQ^2 \cos 2\alpha' = - OR^2 \cos 2\alpha'' \quad (1)$$

and

$$OP^2 \sin 2\alpha + OQ^2 \sin 2\alpha' = - OR^2 \sin 2\alpha''. \quad (2)$$

Now  $P$  and  $Q$  being points on the ellipse, we have from known properties,

$$OP^2 \cos^2 \alpha + OQ^2 \cos^2 \alpha' = a^2, \quad OP^2 \sin^2 \alpha + OQ^2 \sin^2 \alpha' = b^2;$$

hence,  $OP^2 \cos 2\alpha + OQ^2 \cos 2\alpha' = a^2 - b^2$ , and (1) becomes

$$- OR^2 \cos 2\alpha'' = a^2 - b^2. \quad (3)$$

Also, from known properties concerning the ends of conjugate diameters,

$$OP^2 \sin 2\alpha = - OQ^2 \sin 2\alpha';$$

hence, (2) becomes

$$- OR^2 \sin 2\alpha'' = 0. \quad (4)$$

It follows from (3) and (4), that  $2\alpha'' = 180^\circ$  or  $540^\circ$ , and  $OR^2 = a^2 - b^2$ , that is,  $OR = \sqrt{a^2 - b^2}$ , the distance from the center to focus, and  $\alpha'' = 90^\circ$  or  $270^\circ$ , which shows that  $OR$  lies on the minor axis.

Also solved by H. HALPERIN, A. M. HARDING, and H. L. OLSON.

**2780 [1919, 311]. Proposed by ELMER LATSHAW, West Philadelphia, Pa.**

A quadrilateral whose sides are  $a, 2a, 3a, 4a$  is inscribed in a circle. Find the radius of the circle.

### I. SOLUTION BY H. S. UHLER, Yale University.

The interest in this problem may be enhanced by giving a perfectly general solution. Let the sides of any convex inscriptible quadrilateral be denoted by  $a_1, a_2, a_3, a_4$ . A diagonal  $c$  may be drawn dividing the quadrilateral into two non-overlapping triangles the sides of which are  $a_1, a_2, c$  and  $a_3, a_4, c$ , respectively. If the angle between  $a_1$  and  $a_2$  be symbolized by  $C$ , the angle between  $a_3$  and  $a_4$  must be  $180^\circ - C$ . Accordingly

$$c^2 = a_1^2 + a_2^2 - 2a_1a_2 \cos C,$$

$$c^2 = a_3^2 + a_4^2 + 2a_3a_4 \cos C.$$

Eliminating  $2\cos C$  we find

$$c^2 = \frac{(a_1a_3 + a_2a_4)(a_2a_3 + a_4a_1)}{a_1a_2 + a_3a_4}. \quad (1)$$

The area of a plane triangle having the sides  $a_1, a_2, c$  is given by either member of the following equation

$$\frac{a_1a_2c}{4R} = \sqrt{s(s - a_1)(s - a_2)(s - c)}, \quad (2)$$

where  $2s = a_1 + a_2 + c$ , and  $R$  denotes the radius of the circumscribed circle.

Substituting the trinomial value of  $s$  in equation (2) we obtain

$$\frac{a_1a_2c}{R} = \sqrt{[(a_1 + a_2)^2 - c^2][c^2 - (a_1 - a_2)^2]}. \quad (3)$$

Replacing  $c$  in equation (3) by expression (1) we eventually find that

$$R = \frac{\sqrt{(a_1a_2 + a_3a_4)(a_1a_3 + a_4a_2)(a_2a_3 + a_4a_1)}}{\sqrt{(a_2 + a_3 + a_4 - a_1)(a_3 + a_4 + a_1 - a_2)(a_4 + a_1 + a_2 - a_3)(a_1 + a_2 + a_3 - a_4)}}, \quad (4)$$

or

$$R = \frac{1}{4K} \sqrt{(a_1a_2 + a_3a_4)(a_1a_3 + a_4a_2)(a_2a_3 + a_4a_1)}. \quad (5)$$

where if  $2S = a_1 + a_2 + a_3 + a_4$ ,  $K = \sqrt{(S - a_1)(S - a_2)(S - a_3)(S - a_4)}$  = area of quadrilateral.